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GENERIC STRUCTURES AND CONTROL FUNCTIONS (A COMMENTARY ON EVANS'PREPRINT)

東海大学理学部数学科 米田郁生 (IKUO YONEDA)
DEPARTMENT OF MATHEMATICS, TOKAI UNIVERSITY

ABSTRACT. We survey the results in "Some remarks on generic structures" [E] written by Evans, and give some detailed proofs which are omitted in his note.

1. INTRODUCTION

In simplicity theory, Hrushovski's generic constructions yield various results. As in his ω-categorical stable pseudoplane, he constructed an ω-categorical, simple, rank one, non-locally modular theory by amalgamating finite graphs whose local rank is controlled by an increasing unbounded convex function. In [E1], Evans gave a sufficient condition on control functions for constructing ω-categorical simple generic structures. We review this in fifth section. In [E], Evans gave an ω-categorical non-simple generic structure by carefully setting a control function (In this note, sixth section). This non-simple generic structure has 3-strong order property. For any $n \geq 3$, n-strong order property was introduced by Shelah. (See [Sh] and third section in this note.) Strict order property implies n-strong oder property, and $n+1$-strong order property implies n-strong order property for any $n \geq 3$. Evans showed that generic structures given by control functions do not have 4-strong order property, we follow this result in fourth section.
In [P], Pourmahdian conjectured that generic stuructures without control function, so-called $(\mathbf{K}_0, <)$-generic structure, will be non-simple. In [P], Pourmahdina considered a natural expansioned inductive (incomplete) theory T_{nat} of a universal theory T_0 only axiomatizing that any finite substructure has non-zero positive local rank. Pourmahdian showed that T_{nat} is a Robinson theory and its universal domain is simple as a structure, and T_{nat} does not have model companion. (Natural expansioned structure of $(\mathbf{K}_0, <)$-generic structure is an existentially closed model of T_{nat}.) Evans gave an example of $(\mathbf{K}_0, <)$-generic structure having strict order property, we discuss this issue in second section.

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This note is organized as follows.
Section 2: We will follow the proof that $\mathrm{Th}(M_0)$ has strict order property, where M_0 is $(\mathbf{K}_0,<)$-generic structure with one ternary relation.
Section 3: Review of [Sh].
Section 4: We will follow the proof that $\mathrm{Th}(M_f)$ does not have SOP_4, where M_f is a $(\mathbf{K}_f,<)$-generic structure and $\mathbf{K}_f$ is the class of finite graph A satisfying with $\delta(A) \geq f(|A|)$ and control function f is a convex increasing unbounded function from $\mathbb{N}$ to $\mathbb{R}$.
Section 5: Review of [E].
Section 6: We will follow the proof that for some control function f, $\mathrm{Th}(M_f)$ has SOP_3, where $\delta(*) = 2|*| - e(*)$.
Section 7,8: Long appendices for Section 6, which are omitted in [E1].

2. $\mathrm{Th}(M_0)$ HAS SOP. (DEFINABLE CORRESPONDENCE BETWEEN GRAPHS AND TERNARY HYPERGRAPHS)

Let $\mathfrak{R}$ be a ternary relation. For finte ternary-hypergraph $\mathfrak{A}$, we define the predimension as follows.

$$\delta(\mathfrak{A}) = |\mathfrak{A}| - |\mathfrak{R}^{\mathfrak{A}}|$$

For finite $\mathfrak{A} \subseteq \mathfrak{B}$ we define a partial order $<$ as follows

$$\mathfrak{A} < \mathfrak{B} \Leftrightarrow \delta(\mathfrak{X}) > \delta(\mathfrak{A})(\mathfrak{A} \subset \forall \mathfrak{X} \subseteq \mathfrak{B}).$$

For possibly infinite $\mathfrak{A} \subseteq \mathfrak{B}$ we define

$$\mathfrak{A} < \mathfrak{B} \Leftrightarrow \mathfrak{X} \cap A < \mathfrak{X}(\forall \mathfrak{X} \subset_\omega \mathfrak{B}).$$

Note that $\mathfrak{A} < \mathfrak{A}$. For possibly infinite $\mathfrak{A} \subseteq \mathfrak{B}$, there exists the $<$-closure $\mathrm{cl}_{\mathfrak{B}}(\mathfrak{A})$ of $\mathfrak{A}$ in $\mathfrak{B}$. $\mathbf{K}_0$ is the class of finite 3-hypergraphs defined by

$$\mathfrak{A} \in \mathbf{K}_0 \Leftrightarrow \emptyset < \mathfrak{A}$$

$\overline{\mathbf{K}_0}$ is the class of 3-hypergraphs whose finite sub-hypergraph is all in $\mathbf{K}_0$.

M_0 denotes the $(\mathbf{K}_0,<)$-generic structure.

Notation 2.1. Let (A,R) be a graph, where R is the binary relation for the graph. We define the following ternary graph $(\mathfrak{H}_A,\mathfrak{R})$.

- $\mathfrak{H}_A = A \cup R^A \cup \{x_A, y_A\}$, where x_A, y_A are new elements.
- $(\mathfrak{R})^{\mathfrak{H}_A} = \{(x_A,y_A,a) : a \in A\} \cup \{(a,b,(a,b)) : (a,b) \in R^A\}$

$(\mathfrak{H}_A,\mathfrak{R})$ is definable in (A,R) with two new constants.

Lemma 2.2. *Let (A,R) be a graph. Then*

(1) $\mathfrak{H}_A \in \overline{\mathbf{K}_0}$.
(2) $\mathfrak{H}_A = \mathrm{cl}_{\mathfrak{H}_A}(x_A,y_A)$.

Proof. Let $\mathfrak{X} \subset_\omega \mathfrak{H}_A$, and let $V(\mathfrak{X})$ be the vertex set of $\mathfrak{X}$. Then $V(\mathfrak{X}) \subseteq A \cup R^A \cup \{x_A, y_A\}$ follows. If $x_A, y_A \in \mathfrak{X}$, then $\delta(\mathfrak{X}) = |V(\mathfrak{X})| - (|V(\mathfrak{X}) \cap R^A| + |V(\mathfrak{X}) \cap A|) > 0$, since $V(\mathfrak{X}) = \{x_A, y_A\} \cup (V(\mathfrak{X}) \cap R^A) \cup (V(\mathfrak{X}) \cap A)$. Otherwise, $\delta(\mathfrak{X}) = |V(\mathfrak{X})| - (|V(\mathfrak{X}) \cap R^A|) > 0$, since $|V(\mathfrak{X}) \cap R^A| > 0$ implies $|V(\mathfrak{X}) \cap A| > 0$.
Let $c \in A$. Then $\delta(c/x_A, y_A) = 0$. So, if $c \notin \mathrm{cl}_{\mathfrak{H}_A}(x_A, y_A)$, then $0 < \delta(c/\mathrm{cl}_{\mathfrak{H}_A}(x_A, y_A)) \leq \delta(c/x_A, y_A) = 0$, a contradiction. Next, let $c = (a, b) \in R^A$. Then $\delta(c/a, b) = 0$. By the above argument, we see $c \in \mathrm{cl}_{\mathfrak{H}_A}(a, b)$. As $a, b \in \mathrm{cl}_{\mathfrak{H}_A}(x_A, y_A)$, we see that $c \in \mathrm{cl}_{\mathfrak{H}_A}(x_A, y_A)$. □

Next, for any symmetric 3-hypergraph having at least two vertices, we construct a graph as follows.

Notation 2.3. Let $(\mathfrak{A}, \mathfrak{R})$ be a symmetric 3-hypergraph having at least two vertices. Fix two vertices $a, b \in \mathfrak{A}$. We define the following graph $G_{(\mathfrak{A},a,b)} = (G_\mathfrak{A}, R)$ as follows.

- $G_\mathfrak{A} = \{c \in \mathfrak{A} : \mathfrak{A} \models \mathfrak{R}(c, a, b)\}$
- $R^{G_\mathfrak{A}} = \{(c, d) \in \mathfrak{A}^2 : \mathfrak{A} \models \mathfrak{R}(c, a, b) \wedge \mathfrak{R}(d, a, b) \wedge \exists x \mathfrak{R}(x, c, d)\}$

$G_{(\mathfrak{A},a,b)} = (G_\mathfrak{A}, R)$ **is definable in** $(\mathfrak{A}, \mathfrak{R})$ **with parameters** $a, b \in \mathfrak{A}$.

Remark 2.4. (1) $\mathfrak{A} \not\simeq \mathfrak{H}_{G_{(\mathfrak{A},a,b)}}$, where $a, b \in \mathfrak{A}$. (If $\mathfrak{A} \models \neg\mathfrak{R}(d, a, b)$, d will not appear in the righthand.)
(2) $A \simeq G_{(\mathfrak{H}_A, x_A, y_A)}$.

Proof. Clearly, $G_{\mathfrak{H}_A} = A$ and $R^{G_{\mathfrak{H}_A}} = R^A$, as desired.

Lemma 2.5. *Let $(\mathfrak{A}, \mathfrak{R})$ be a symmetric 3-hypergraph having at least two vertices. Then*

(1) $a, b \notin G_\mathfrak{A} \subseteq \mathrm{cl}_\mathfrak{A}(a, b)$
(2) *If* $(c, d) \in R^\mathfrak{A}$, *then* $\mathfrak{R}^{\mathrm{cl}_\mathfrak{A}(a,b)} \models \exists x \mathfrak{R}(x, c, d)$
(3) *If* $\mathfrak{A} < \mathfrak{B}$, *then* $G_{(\mathfrak{A},a,b)} = G_{(\mathfrak{B},a,b)}$

Proof. If $\mathfrak{A} \models \mathfrak{R}(c, a, b)$, then $c \in \mathrm{cl}_\mathfrak{A}(a, b)$. (1),(2) follow. If $\mathfrak{A} < \mathfrak{B}$, then $\mathrm{cl}_\mathfrak{A}(a, b) = \mathrm{cl}_\mathfrak{B}(a, b)$. So, (3) follows. □

Notation 2.6. Let φ be a sentence in the language of graphs with binary relation symbol $R(x_1, x_2)$. **We construct a fourmula** σ_φ **having free variable** y, z **in the the language of 3-hypergraphs with ternary relation symbol** $\mathfrak{R}(x_1, x_2, x_3)$ **as follows.**

- Replace all atomic subformulas $R(x_1, x_2)$ by $\mathfrak{R}(x_1, y, z) \wedge \mathfrak{R}(x_2, y, z) \wedge \exists w \mathfrak{R}(w, y, z)$
- Replace $\forall x(\psi(\bar{x})), \exists x(\psi(\bar{x}))$ by $\forall x(\mathfrak{R}(x, y, z) \to \psi(\bar{x})), \exists x(\mathfrak{R}(x, y, z) \wedge \psi(\bar{x}))$.

Remark 2.7. Let $(\mathfrak{A}, \mathfrak{R}) \in \overline{\mathbf{K}_0}$, $a, b \in \mathfrak{A}$ and φ be a sentence in the language of graphs. Then

$$G_{(\mathfrak{A},a,b)} \models \varphi \Leftrightarrow \mathfrak{A} \models \psi_\varphi(a,b)$$

The above remark follows from "REDUCTION THEOREM" , a (non-onto) map from $G_{(\mathfrak{A},a,b)}$ to $\mathfrak{A}$, and the way of replacement of quantifiers. Reduction theorem needs a onto map, but our ψ_φ's quatifiers are bounded in $\mathfrak{R}(*, a, b)$. So we need not a onto map, here.

Fact 2.8. *Let M be an L-structure, and N be an L'-structure. Suppose that*

- *there exists a partial onto map f from M^n to N (for some $n < \omega$)*
- *for every positive atomic L-formula θ, there exists an L'-formula ψ_θ such that $M \models \theta(\bar{a}) \Leftrightarrow N \models \psi_\theta(f(\bar{a}))$*

THEN, by induction on the complexity of formulas, for every L-formula φ, there exists an L'-formula ψ_φ such that $M \models \varphi(\bar{a}) \Leftrightarrow N \models \psi_\varphi(f(\bar{a}))$

Lemma 2.9. *Let φ be a sentence in the language of graphs. THEN, "there exists a finite graph $A \models \varphi$" iff $M_0 \models \exists yz\psi_\varphi(y,z)$.*

Proof. $(\Rightarrow)$: We may assume that $\mathfrak{H}_A < M_0$. So, by Remark 2.4, $A \simeq G_{(\mathfrak{H}_A, x_A, y_A)} \simeq G_{(M_0, x_A, y_A)}$. Therefore, $M_0 \models \psi_\varphi(x_A, y_A)$.
$(\Leftarrow)$: $G_{(M_0,a,b)} \models \varphi$ and $G_{(M_0,a,b)} \subseteq \mathrm{cl}_{M_0}(a,b) \subset_\omega M_0$ □

Proposition 2.10. *Let φ be a sentence in the language of graphs. Suppose that φ has arbitralily large finite model. Then there exists an infinite model, definable in some model of* $\mathrm{Th}(M_0)$.

Proof. By our assumption, for any $n < \omega$, there exists a finite graph A_n such that $A_n \models \varphi$ and $|A_n| \geq n$. As $A_n \simeq G_{(\mathfrak{H}_{A_n}, x_{A_n}, y_{A_n})}$ (by Remark 2.4) and $\omega > |\mathfrak{H}_{A_n}| \geq |A_n| \geq n$, for any $n < \omega$,

$$\mathfrak{H}_{A_n} \models \psi_\varphi(x_A, y_A) \wedge |\mathfrak{R}^{\mathfrak{H}_{A_n}}(*, x_A, y_A)| \geq n.$$

As M_0 is $(\mathbf{K}_0, <)$-generic, there exists $\mathfrak{H}_{A_n} \simeq \mathfrak{A} < M_0$. Since $G_{(\mathfrak{H}_{A_n}, x_{A_n}, y_{A_n})} \simeq G_{(\mathfrak{A},a,b)} = G_{(M,a,b)}$, where $x_A y_A \mapsto ab$,

$$\mathrm{Th}(M_0) \vdash \exists yz\psi_\varphi(y,z) \wedge |\mathfrak{R}(*, y, z)| \geq n.$$

By compactness, there exist infinite $M \models \mathrm{Th}(M_0)$ and $a', b' \in M$ such that $G_{(M,a',b')} \models \varphi$, where $G_{(M,a',b')}$ is definable in M. □

Theorem 2.11. $\mathrm{Th}(M_0)$ *has strict order property.*

Proof. Let A_n be the graph as follows;

- Vertices: $\{b_i : i < n\} \cup \{c_i : i < n\}$
- Edges: $\{(b_i, c_j) : 0 \leq i < j < n\}$

Let $a_i = (b_i, c_i)$, and $\varphi(xy, zw) \equiv R(x,y) \wedge R(z,w) \wedge R(x,w) \wedge \neg R(x,z) \wedge \neg R(y,w) \wedge \neg R(y,z)$. Then $A_n \models \varphi(a_i, a_j) \Leftrightarrow i < j < n$.

By Lemma 2.9, we can find a linear (uniformly definable) ordering of arbitraily finite length in M_0. By compactness, we see that $\mathrm{Th}(M_0)$ has the strict order property. □

3. Review of Strong Order Property

This section consists of Shelah's results in [Sh].

Definition 3.1. A complete theory T has n-strong order property, denoted SOP_n if there exists a formula $\varphi(x,y)$ ($\mathrm{lh}(x) = \mathrm{lh}(y)$) and a sequence $(a_i : i < \omega)$ in some model N of T such that

(1) $N \models \varphi(a_i, a_j)$ for $i < j < \omega$

(2) there is no n-φ-loops;

$$N \models \neg \exists x_0, x_1, \dots x_{n-1} \varphi(x_0, x_1) \wedge \varphi(x_1, x_2) \wedge \cdots \wedge \varphi(x_{n-2}, x_{n-1})$$

Fact 3.2. (1) SOP *implies* SOP_n.

(2) SOP_{n+1} *implies* SOP_n.

(3) *If* T *has* SOP_3*, then* T *has the tree property.*

Proof. (1): By way of contradiction, suppose that T has SOP and $NSOP_n$. So, there exist $\varphi(x,y)$, $N \models T$ and $(a_i : i < \omega) \subset N$ such that $\forall x(\varphi(x,a_i) \rightarrow \varphi(x,a_j)) \wedge \exists x(\neg\varphi(x,a_i) \wedge \varphi(x,a_j))$ for $i < j < \omega$. Let $\psi(x_0, x_1) = \forall x(\varphi(x,x_0) \rightarrow \varphi(x,x_1)) \wedge \exists x(\neg\varphi(x,x_0) \wedge \varphi(x,x_0))$. As T has $NSOP_n$, there exists n-ψ-loop, but it is impossible.

(2): Let $\varphi(x,y)$, a model M, and $(a_i : i < \omega) \in M$ be witness for SOP_{n+1}. We may assume that $(a_i : i < \omega)$ is indiscernible. We divide the arugument into two cases, whether

$$M \models \exists x_0, \dots x_{n-1}[x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j<n, k \equiv l+1 (\mathrm{mod} n)} \varphi(x_i, x_j)$$

or not.

- The case that $M \models \exists x_0, \dots x_{n-1}[x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j<n,k\equiv l+1(\mathrm{mod} n)} \varphi(x_i, x_j)]$

As $a_1 \equiv_{a_0} a_2$, we have $M \models \exists x_0, \dots x_{n-1}[x_0 = a_2 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j<n,k\equiv l+1(\mathrm{mod} n)} \varphi(x_i, x_j)]$. Let $a_2, c_1, \dots c_{n-2}, a_0$ be the witness for $x_0, \dots x_{n-1}$. By the way, $M \models \varphi(a_1, a_2) \wedge \varphi(a_0, a_1)$, so $a_1, a_2, c_1, \dots c_{n-2}, a_0$ is an $(n+1)$-φ-loop, a contradiction.

- The case that $M \not\models \exists x_0, \dots x_{n-1}[x_0 = a_1 \wedge x_{n-1} = a_0 \wedge \bigwedge_{i,j<n,k\equiv l+1(\mathrm{mod} n)} \varphi(x_i, x_j)]$

Put $\psi(x,y) \equiv \varphi(x,y) \wedge \neg \exists x_0, \dots x_{n-1}[x_0 = x \wedge x_1 = y] \wedge \bigwedge_{i,j<n,k\equiv l+1(\mathrm{mod} n)} \varphi(x_i, x_j)$
Then $M \models \psi(a_i, a_i)(i < j < \omega)$, and n-ψ-loops never exist.

(3): Let $\kappa = \mathrm{cf}(\kappa) > |T|$ and $\lambda > \kappa$ be such that $\mathrm{cf}(\lambda) = \kappa$ and "$\mu < \lambda$ implies $2^\mu < \lambda$" (strongly limit singular cardinal of cofinality κ). Put $J = {}^\kappa\lambda$ and

$I \subset J$ be such that $\eta \in I$ iff $\eta(i) = 0$ for every $i < \kappa$ large enough.
Let $\varphi(x, y)$ be the witnee for SOP_3. By compactness, there exist a sequence $(a_\eta : \eta \in I)$ in some model M shch that $M \models \varphi(a_\eta, a_\nu)$ for any $\eta < \nu$.
The lexicographic order on I is as usual; if i is the least such that $\eta|i = \nu|i$, then $\eta(i) = \nu(i)$.
We may assume that M is κ^+-saturated, and $|M| \geq \lambda$. Fix an $\eta \in {}^\kappa(\lambda \backslash \{0\}) \backslash I$. We will define a_η as follows.
Put $p_\eta = \{\varphi(a_{(\eta|i)0_{[i,\kappa)}}, x) \wedge \varphi(a_{(\eta|i,\eta(i)+1)0_{(i,\kappa)}}, x) : i < \kappa\}$.
Note that $(\eta|i)0_{[i,\kappa)}, (\eta|i, \eta(i)+1)0_{(i,\kappa)} \in I$, and
$a_{(\eta|i\eta(i))0_{(i,\kappa)}} \models \varphi(a_{(\eta|i)0_{[i,\kappa)}}, x) \wedge \varphi(a_{(\eta|i\eta(i)+1)0_{[i,\kappa)}}, x)$.
As M is κ^+-saturated, there exists a realization of p_η in M, say a_η.

Claim. *If $\eta_1 \neq \eta_2 \in {}^\kappa(\lambda \setminus \{0\})$, then $p_{\eta_1} \cup p_{\eta_2}$ is inconsitent.*

Suppose that $\eta_1 < \eta_2$. Then there exists $i < \kappa$ such that $\eta_1|i = \eta_2|i, \eta_1(i) < \eta_2(i)$. Take $\nu < \rho \in I$ be with $\eta_1 < \nu < \rho < \eta_2$ as follows.
$\eta_1|i = \eta_2|i = \nu|i = \rho|i, \nu(i) = \eta_1(i) + 1, \rho(i) = \nu_2(i), \nu(j) = 0 (j > i), \rho(i+1) = \nu_2(i+1), \rho(j) = 0 (j > i+1)$.
As $\varphi(x, a_\nu) \in p_{\eta_1}, \varphi(x, a_\rho) \in p_{\eta_2}$,and $M \models \varphi(a_\nu, a_\rho)$, if we found the realization of $p_{\eta_1} \cup p_{\eta_2}$, say c, then c, a_ν, a_ρ would be the 3-φ-loop, a contradiction.

We also have $|p_\eta| = \kappa, |\bigcup\{\mathrm{Dom}(p_\eta) : \eta \in {}^\kappa(\lambda \setminus \{0\})\}| \leq \lambda$ (as $\bigcup\{\mathrm{Dom}(p_\eta) : \eta \in {}^\kappa(\lambda \setminus \{0\})\} \subseteq \{a_\nu : \nu \in I\}$)

By 7.7(3) and 7.6(2) on p.141 of Shelah's 2nd editon book, $\lambda = \lambda^{<\kappa} > 2^{|T|}$ (by $\mathrm{cf}(\lambda) = \kappa < \lambda$) and $\kappa > |T|$ imply that T has the tree property. $\square$

It is conjectured that SOP_4 is a good dividing line for existence of universal models, i.e. if T does not have SOP_4, it will have universal models of cardinality $\lambda > |T|$ (Shelah showed that if T is simple and $\lambda > |T|$, then there exists universal models of cardinality λ^{++}. As the above, simplicity implies $N\mathrm{SOP}_3$.)

4. $\mathrm{Th}(M_f)$ DOES NOT HAVE SOP_4

Let δ be a local rank on relational finite structures such that $\delta(A/B) \leq \delta(A/A \cap B)$, where $\delta(A/B) = \delta(AB) - \delta(B)$. Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be upper unbounded and monotone increasing. Let $\mathbf{K}_f = \{A \in \mathbf{K}_0 : \delta(X) \geq f(|X|)(\forall X \subseteq A)\}$ and $\beta(x) = \min\{\delta(X/A) : A < X \in \mathbf{K}_0, A \neq X, |X| \leq x\}$.

Fact 4.1. *Suppose that*

$$f'(x) \leq \frac{\beta(x)}{x}.$$

Then $\mathbf{K}_f$ *is closed under free amalgamation, so* $(\mathbf{K}_f, <)$*-generic* M_f *exists,* $\mathrm{cl} = \mathrm{acl}$ *in* M_f *and* $\mathrm{Th}(M_f)$ *is* ω*-categorical. (*ω*-categoricity follows from* $|\mathrm{cl}(*)| \leq f^{-1}(\delta(*))$ *for finite graphs.)*

Proof. Let $A < B_1, B_2 \in \mathbf{K}_f$ and let $C = B_1 \otimes_A B_2$. We need to show that if $X \subseteq C$, then $\delta(X) \geq f(|X|)$. We may assume that $X < C$, because $\delta(X) \geq \delta(\mathrm{cl}(X))$ and $f(|\mathrm{cl}(X)|) \geq f(|X|)$.

Let $X_i = X \cap B_i (i = 1, 2)$ and let $X_0 = X \cap A$. Suppose that

$$\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}.$$

As $X_0 < X_1$, $\beta(|X_1|) \leq \delta(X_1/X_0)$. Therefore $\dfrac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \geq \dfrac{\beta(|X_1|)}{|X_1|} \geq$ $f'(|X_1|)$. So, the line between $(|X_0|, \delta(X_0))$ and $(|X_1|, \delta(X_1))$ lies above f. As f' is decreasing and $\delta(X_1) \geq f(|X_1|)$, $\delta(X) \geq f(|X|)$ follows. □

Let $d(A) = \delta(\mathrm{cl}(A))$, and $d(a/A) = \delta(\mathrm{cl}(aA)/\mathrm{cl}(A))$. For possibly infinite B, let $d(a/B) = \inf\{d(a/B_0) : B_0 \subset_\omega B\}$.

Fact 4.2. *Let* $\mathcal{M}$ *be a relational structure having* δ*-rank. Let* $a, A, B \subset_\omega \mathcal{M}$*. Suppose that* $A < B < \mathcal{M}$ *and* $\mathrm{cl}(aA) \subset_\omega \mathcal{M}$*. Then* $d(a/B) = d(a/A)$ *iff* $\mathrm{cl}(aA) \cap B = A, \mathrm{cl}(aA)B = \mathrm{cl}(aA) \otimes_A B$ *and* $d(aB) = \delta(\mathrm{cl}(aA)B)$ *(i.e.* $\mathrm{cl}(aA)B \leq \mathrm{cl}(aB)$*).*

Proof. As $A < \mathrm{cl}(aA) \cap B$ or $A = \mathrm{cl}(aA) \cap B$, we have $\delta(A) \leq \delta(\mathrm{cl}(A) \cap B)$. So, $\delta(\mathrm{cl}(aA)/\mathrm{cl}(aA) \cap B) \leq \delta(\mathrm{cl}(aA)/A)$. Therefore

$$d(a/B) \leq \delta(\mathrm{cl}(aA)/B) \leq \delta(\mathrm{cl}(aA)/\mathrm{cl}(aA) \cap B) \leq \delta(\mathrm{cl}(aA)/A) = d(a/A).$$

Now we can see the conclusion. □

By Fact 4.2, for $a, b, A \subset_\omega \mathcal{M}$,

$$d(a/Ab) = d(a/A) \Leftrightarrow d(b/Aa) = d(b/A).$$

(By $d(a/Ab) = d(a/A) \Leftrightarrow$ "$\mathrm{cl}(aA) \cap \mathrm{cl}(bA) = \mathrm{cl}(A), \mathrm{cl}(aA)\mathrm{cl}(bA) = cl(aA) \otimes_{\mathrm{cl}(A)} \mathrm{cl}(bA) \leq \mathrm{cl}(abA)$".)

From now on, we assume that the control function f saitsfies "$f'(x) \leq \dfrac{\beta(x)}{x}$". Let $\overline{\mathbf{K}_f}$ be the class of possibly infinite structures whose finite substructures are all in $\mathbf{K}_f$. Let $T_f = \{\forall \bar{x} \neg \mathrm{Diag}_A(\bar{X}) : \delta(A) < f(|A|), |A| < \omega\}$. Then $M \models T_f \Leftrightarrow M \in \overline{\mathbf{K}_f}$. Let $\mathcal{M}$ be a big model of M_f. Note that if $A \subset_\omega \mathcal{M}$, then $A \in \mathbf{K}_f$.

Proposition 4.3. *Suppose that, in* $\mathcal{M}$*, if* $A = \mathrm{acl}(A)$*,* $d(a/A) = d(a/Ab)$*,* $\mathrm{acl}(aA) \cap \mathrm{acl}(bA) = A$*, then there exists* $A_0 \subset_\omega A$ *such that* $d(a/A_0 b) = d(a/A_0)$*. THEN* $\mathrm{Th}(M_f)$ *has* $N\mathrm{SOP}_4$*.*

Proof. Let $(a_i : i < \omega)$ be an infinite indiscernible sequence in $\mathcal{M}$. Put $p(x_0x_1) = \mathrm{tp}(a_0a_1)$. We will show that

$$p(x_0x_1) \cup p(x_1, x_2) \cup p(x_2x_3) \cup p(x_3x_0)$$

is consistent.

Claim. *There exists $B \subset_\omega \mathcal{M}$ such that $(a_i : i < \omega)$ is B-indiscernible, and $d(a_2/Ba_0a_1) = d(a_2/Ba_1) = d(a_2/B)$. (Then $a_2 \equiv_{a_0} a_1$, $d(a_2/Ba_0a_1) = d(a_2/Ba_2)$ follows.)*

Extend $(a_i : i < \omega)$ to $(a_i : i < \mathbb{Z})$. As $(a_i : i \geq 0)$ is indiscernible over $(a_i : i < 0)$, $(a_i : i \geq 0)$ is indiscernible over $\mathrm{acl}(a_i : i < 0) =: A_0$. As $a_{<i} \equiv_{a_i} a_{<0}$, we see that $d(a_i/A_0a_{<i}) = d(a_i/A_0)$.

By extending $(a_i : i \geq 0)$ over A_0 and applying Erdos-Rado Theorem, we may assume that $\mathrm{acl}(A_0a_k) \cap \mathrm{acl}(A_0a_ia_j) =: C$ is constant for any $i < j < k$, and $(a_i : i \geq 0)$ is indiscernible over C.

Now, by our assumption, take $B \subset_\omega C$ such that $d(a_2/Ba_0a_1) = d(a_2/B)$, as desired. The claim is proven.

As $d(a_2/Ba_0a_1) = d(a_2/B)$, we have

$$\mathrm{cl}(a_2B)\mathrm{cl}(a_0a_1B) = \mathrm{cl}(a_2B) \otimes_{\mathrm{cl}(B)} \mathrm{cl}(a_0a_1B) \leq \mathrm{cl}(a_0a_1a_2B).$$

As $\mathrm{cl}(a_0a_1a_2B) \in \mathbf{K}_f$, we may assume that

$$\mathrm{cl}(a_0a_1a_2B) < M_f.$$

So, we can work inside M_f.(i.e. we have $a_0, a_1, a_2, B \subset_\omega M_f$ such that (a_0, a_1, a_2) is B-indiscernible and $d_{M_f}(a_2/Ba_0a_1) = d_{M_f}(a_2/B)$.)

Let $C_{i,j} = \mathrm{cl}(a_ia_jB), C_i = \mathrm{cl}(a_iB)$. By $d(a_2/Ba_0a_1) = d(a_2/Ba_1)$ and Fact 4.2, we see that $C := C_{0,1}C_{1,2} = C_{0,1} \otimes_{C_1} C_{1,2}$. And $C_{0,1} \cap C_{0,2} = C_0$ and $C_{1,2} \cap C_{0,2} = C_2$ follow by $d(a_2/Ba_0a_1) = d(a_2/Ba_1), d(a_1/Ba_0a_2) = d(a_1/Ba_2)$ and Fact 4.2. So we have

$$C \cap C_{0,2} = C_0C_2 = C_0 \otimes_B C_2 < C.$$

Let $f : C_0C_2 \to C_2C_0$ be an isomorphism over B sending a_0a_2 to a_2a_0, and let $g : C_0C_2 \hookrightarrow C$ be the inclusion map. Put $g' = g \circ f$. As $\mathbf{K}_f$ is closed under free amalgamation, there exist $D \in K_f$ and $h, h' : C \hookrightarrow D$ such that $h \circ g|C_0C_2 = h' \circ g'|C_0C_2$ and $D = h(C) \otimes_{h \circ g(C_0C_2)} h'(C)$. We may assume that $D < M_f$. Put $a_0' = h \circ g(a_0), a_1' = h(a_1), a_2' = h' \circ g'(a_2), a_3' = h'(a_1)$.

Claim. $a_0'a_1', a_1'a_2', a_2'a_3', a_3'a_0' \models p = \mathrm{tp}(a_0a_1)$. *(This proposition is proven.)*

Note that

$$h(a_0a_1) = a_0'a_1', h(a_1a_2) = a_1'a_2', h'(a_0a_1) = (h' \circ g'(a_2))a_3' = a_2'a_3',$$
$$h'(a_1a_2) = a_3'(h' \circ g'(a_0)) = a_3'(h \circ g(a_0)) = a_3'h(a_0) = a_3'a_0'.$$

On the other hand,

$$h(C_{0,1}), h(C_{1,2}) < h(C) < D < M_f,$$
$$h'(C_{0,1}), h'(C_{1,2}) < h'(C) < D < M_f.$$

Put $B' = h \circ g(B) = h' \circ g'(B)$. Then

$$h(\mathrm{cl}(a_0a_1B)) = h(C_{0,1}) = \mathrm{cl}(a'_0a'_1B'), h(\mathrm{cl}(a_1a_2B)) = h(C_{1,2}) = \mathrm{cl}(a'_1a'_2B'),$$
$$h'(\mathrm{cl}(a_0a_1B)) = h'(C_{0,1}) = \mathrm{cl}(a'_2a'_3B'), h(\mathrm{cl}(a_1a_2B)) = h(C_{1,2}) = \mathrm{cl}(a'_3a'_0B').$$

By genericity of M_f, we see that

$$\mathrm{cl}(a_0a_1B) \equiv \mathrm{cl}(a_1a_2B) \equiv \mathrm{cl}(a'_0a'_1B') \equiv \mathrm{cl}(a'_1a'_2B') \equiv \mathrm{cl}(a'_2a'_3B') \equiv \mathrm{cl}(a'_3a'_0B').$$

□

Remark 4.4. Suppose that for any $a, A \subset \mathcal{M}$, there exists $A_o \subset_\omega A$ such that $d(a/A) = d(a/A_0)$. Then the assumption of Proposition 4.3 holds.

Proof. Take $A_0, A_1 \subset_\omega A$ such that $d(a/Ab) = d(a/A_0b)$ and $d(a/A) = d(a/A_1)$. Then $d(a/A_0A_1) = d(a/A_0A_1b)$. □

5. Review of Evans' paper on simple ω-categorical generic structures

Let δ be a local rank on relational finite structures such that $\delta(A/B) \leq \delta(A/A \cap B)$, where $\delta(A/B) = \delta(AB) - \delta(B)$. Let $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be upper unbounded, monotone increasing, convex ($f'(x)$ is monotone decreasing) and $f'(x) \leq \dfrac{\beta(x)}{x}$, where $\beta(x) = \min\{1, \delta(X/A) : A < X \in \mathbf{K}_0, A \neq X, |X| \leq x\}$. Let $\mathbf{K}_f = \{A \in \mathbf{K}_0 : \delta(X) \geq f(|X|)(\forall X \subseteq A)\}$.

The following fact is Corollary 2.20 of [E1].

Fact 5.1. *Let M_f be $(\mathbf{K}_f, <)$-generic. And suppose the condition on $\mathcal{M}$ (big model of* $\mathrm{Th}(M_f)$*) as in Proposition 4.3. Furthermore, suppose the following.*

(1) *(d-extension property in $\mathcal{M}$)*
Let $A \subset B \subset \mathcal{M}$ be algebraically closed and $c \subset_\omega \mathcal{M}$. Then there exists $c' \subset_\omega \mathcal{M}$ such that $\mathrm{tp}(c/A) = \mathrm{tp}(c'/A)$, $d(c'/B) = d(c/A)$ *and* $\mathrm{acl}(c'A) \cap B = A$.
(2) *(Independence theorem over finite closed sets in M_f)*
Let $A, B_1, B_2 < M_f$ be finite such that $B_1 \cap B_2 = A$ and $d(B_1/B_2) = d(B_1/A)$. Suppose that $c_1, c_2 \subset_\omega M_f$, $\mathrm{tp}(c_1/A) = \mathrm{tp}(c_2/A)$ *and* $d(c_i/B) = d(c_i/A)$. *then there exists $c \subset_\omega M_f$ such that* $\mathrm{tp}(c/B_i) = \mathrm{tp}(c_i/B_i)$ *and* $d(c/B_1B_2) = d(c/A)$.

THEN $\mathrm{Th}(M_f)$ *is simple and "$c \mathop{\smile\!\!\!\!\!\!\,|}_A B \Leftrightarrow d(c/B) = d(c/A)$ and* $\mathrm{acl}(cA) \cap B = A$, *for A, B algebraically closed in $\mathcal{M}$".*

We give the proof of the following lemma.(Theorem 3.6 of [E1])

Lemma 5.2. *Supose that d-extension property over finite closed sets in* $\mathcal{M}$ *and* $f(3x) \leq f(x) + \beta(x)$. *Then the independence theorem over finite closed sets holds in* M_f.

Proof. Let c_i, B_i, A be as in Fact 5.1. Then $\mathrm{acl}(c_1A) \simeq_A \mathrm{acl}(c_2A)$. Put $E_{12} = \mathrm{acl}(B_1B_2), E_{13} = \mathrm{acl}(c_1B_1), E_{23} = \mathrm{acl}(c_2B_2)$. By considering free amalgamation and copies, we may assume that
$B_1 = E_{12} \cap E_{13}, B_2 = E_{12} \cap E_{23}, B_3 := E_{13} \cap E_{23} = \mathrm{acl}(c_iA)$,
$B_1 \cap B_2 \cap B_3 = A$, B_1, B_2, B_3 are d-independent over A, $E_{ij}E_{jk} = E_{ij} \otimes_{B_j} E_{jk}$.
Let $E = E_{12}E_{13}E_{23}$. We need to show that $A < E$ and $E \in \mathbf{K}_f$.

Claim. $A < E$.

By Fact 4.2, $B_iB_j \leq E_{ij}$. As $E = E_{ij} \otimes_{B_iB_j} E_{ik}E_{jk}$, $E_{ik}E_{jk} \leq E$ follows. We also have $E_{ik}E_{jk} = E_{ik} \otimes_{B_k} E_{jk}$ and $B_k < E_{jk}$, $E_{ik} < E_{ik}E_{jk}$ follows. Thus $E_{ik} < E$. As $A < B_i < E_{ik}$, $A < E$ follows.

Claim. $E \in \mathbf{K}_f$.

We have $E = E_{ij} \otimes_{B_iB_j} E_{ik}E_{jk}$, but we do not have $B_iB_j < E_{ij}, E_{ik}E_{jk}$. So we can not conclude this claim by using Fact 4.1.
We need to show $\delta(D) \leq f(|D|)$ for any $D < E$ as in Fact 4.1.Put $D_{ij} = D \cap E_{ij}$ and $d_{ij} = \delta(D_{ij})$. Suppose that d_{12} is the largest of these.
As $E_{12}E_{23} \in \mathbf{K}_f$, we may assume that $D \neq D_{12}D_{23}$. Put $D^1 = D_{12}D_{13}$. As $E_{12}E_{13} \leq E$, we see that $D^1 \leq D$. As $D^1 = D_{12} \otimes_{D \cap B_1} D_{13}$ and $D \cap B_1 < D_{13}$,

$$\delta(D^1) = d_{12} + \delta(D_{13}/D \cap B_1) \geq d_{12} + \beta(|D_{13}|).$$

As $d_{13} \leq d_{12}$, $|D_{13}| \leq f^{-1}(d_{13}) \leq f^{-1}(d_{12})$.
So, as β is monotone decreasing, $d_{12} \leq \delta(D^1) - \beta(|D_{13}|) \leq \delta(D^1) - \beta(f^{-1}(d_{12}))$.
By our assumption on f
$(f(3x) \leq f(x) + \beta(x)$, so $3f^{-1}(x) \leq f^{-1}(x + \beta(f^{-1}(x)))$,

$$3f^{-1}(d_{12}) = f^{-1}(d_{12} + \beta(f^{-1}(d_{12}))).$$

So, $3f^{-1}(d_{12}) \leq f^{-1}(\delta(D^1))$. As $|D| \leq \sum_{ij} |D_{ij}| \leq \sum_{ij} f^{-1}(d_{ij}) \leq 3f^{-1}(d_{12})$ and $\delta(D^1) \leq \delta(D)$, we see that

$$|D| \leq f^{-1}(\delta(D)).$$

□

6. Th(M_f) HAS SOP$_3$ FOR SOME f

We work with undirected graphs, and $\delta(A) = 2|A| - e(A)$. Note that $\beta(x) = 1$. The control fuction $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is an upper unbounded, monotone increasing satisfying the following five conditions;

(F1): $f(0) = 0, f(2) = 2, f(4) = 3, f(8) = 4 < f(10) < 4\frac{1}{2} < f(12) < 5 < f(14) < 5\frac{1}{3} < f(16) < f(18) \leq 6$.

(F2): $2f'(2n) \leq \frac{1}{n}$ for $n \geq 7$

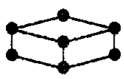
(F3): $f(\frac{k^2}{2}) \leq k$ if $k \geq 6$

(F4): $f(3n) \leq f(n) + 1$ for $n \geq 10$.

(F5): $f(10) + 1 \geq f(14), f(12) + 1 \geq f(16)$.

Let $f_1(x) = f(2x)$. So, $f_1'(x) = 2f'(2x)$ and F2: $f_1'(n) \leq \frac{1}{n}$ for $n \geq 7$. We consider K_{f_1}.

Remark 6.1. (1) δ(3-cycle)$= 6 - 3 = 3 = f(4) < f(6) = f_1(3)$, so 3-cycle does not belong to K_{f_1}. δ(4-cycle)$= 8 - 4 = 4 = f(8) = f_1(4)$, so 4-cycle belongs to K_{f_1}.

(2) The graph  does not belong to $\mathbf{K}_{f_1}$, because its δ-rank$= 14 - 9 = 5 < f(14) = f_1(7)$.

(3) (F1) and (F2) give the free amalgamation property of $(\mathbf{K}_{f_1}, <)$.

(4) (F1) and (F3) are needed to show that the graphs $G(A_n, B_n, x_0)$ belong to $\mathbf{K}_{f_1}$. (Lemma 6.4.)

(5) (F4) is needed to show Subclaim 2 in the proof of Lemma 6.7. Lemma 6.7 ensures that the important graphs E_n can be closedly embedded into M_{f_1} and the graphs E_n will give the witness formula for SOP_3.

(6) (F1), (F2) and (F5) are needed to show Lemma 6.6. (Lemma 6.6 gives a very important key to get Lemma 6.7.)

By the graphs $E_n < M_{f_1} (n \in \omega)$, we will give a formula $\varphi(x, y)$ and infinite sequence $(a_i)_{i<\omega}$ in M_{f_1} such that $M_{f_1} \models \varphi(a_i, a_j)$ whenever $i < j$. But if there were a 3-φ-loop in some model N of Th(M_{f_1}) , then N would have the graph as in (2) of Remark 6.1. As any finite graph of N belongs to $\mathbf{K}_{f_1}$, so SOP_3 follows.

Lemma 6.2. $\mathbf{K}_{f_1}$ *has the free amalgamation property.*

Proof. Let $A < B_1, B_2 \in \mathbf{K}_f$ and let $C = B_1 \otimes_A B_2$. We need to show that if $X \subseteq C$, then $\delta(X) \geq f_1(|X|)$. We may assume that $X < C$, because $\delta(X) \geq \delta(\mathrm{cl}(X))$ and $f_1(|\mathrm{cl}(X)|) \geq f_1(|X|)$.

Let $X_i = X \cap B_i (i = 1, 2)$ and let $X_0 = X \cap A$. Suppose that

$$\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}, |X_1| \geq 7.$$

As $X_0 < X_1$, $\beta(|X_1|) \leq \delta(X_1/X_0)$. So, by (F2), $\dfrac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \geq \dfrac{1}{|X_1|} \geq f_1'(|X_1|)$. So, the line between $(|X_0|, \delta(X_0))$ and $(|X_1|, \delta(X_1))$ lies above f_1. As f_1' is decreasing and $\delta(X_1) \geq f_1(|X_1|)$, $\delta(X) \geq f_1(|X|)$ follows. In Appendix 1, we give the proof when $|X_1| \leq 6$.

□

Notation 6.3. Consider the following graphs $G(A_n, B_n, x_0)$ for each $n < \omega$.

- Vertex set: $A_n \cup B_n \cup \{x_0\} \cup \{z_{ij} : 0 \leq i < j \leq n\}$, where $A_n = \{a_i : 0 \leq i \leq n\}, B_n = \{b_i : 0 \leq i \leq n\}$.
- Edges: $R(x_0, a_i), R(x_0, b_i)$ for $0 \leq i \leq n$ and $R(z_{ij}, a_i), R(z_{ij}, b_j)$ for $0 \leq i < j \leq n$.

Lemma 6.4. (1) $G(A_n, B_n, x_0) \in \mathbf{K}_{f_1}$
(2) $x_0 A_n < G(A_n, B_n, x_0)$
(3) $d(A_n/B_n) = d(A_n/x_0)$, *where* $d(*) = d_{G(A_n,B_n,x_0)}(*)$.

Proof. Put $G = G(A_n, B_n, x_0), A = A_n, B = B_n, Z = \{z_{ij} : 0 \leq i < j \leq n\}$.
(1): It suffices to show that if $X < G$, then $\delta(X) \geq f_1(|X|)$. It is clear in case of $|X| = 1$. If $|X| \geq 2$, then $x_0 \in X$. (If $x_0 \neq a, b \in X$, then $\delta(x_0/ab) = 0$, so $x_0 \in \mathrm{cl}_G(ab) \subset X$.)

Claim. $a_i, b_j \in X \Leftrightarrow z_{ij} \in X$.

This claim follows from $\delta(z_{ij}/a_i b_j) = \delta(a_i/x_0 z_{ij}) = \delta(b_j/x_0 z_{ij}) = 0$ and $X < G$.

Put $X_A = X \cap A_n, X_B = X \cap B, X_Z = X \cap Z$ and $m = |X_A| + |X_B|$. By claim, we see that $\delta(X_Z/x_0 X_A X_B) = 0$, so we have

$$\delta(X) = \delta(x_0 X_A X_B) = 2(m+1) - m = m + 2 =: k \dagger$$

As $|X_Z| \leq |X_A||X_B| \leq |X_A|(m - |X_A|) = (\frac{m}{2})^2 - (|X_A| - \frac{m}{2})^2 \leq (\frac{m}{2})^2$, we have

$$|X| \leq 1 + m + (\frac{m}{2})^2 = (1 + \frac{m}{2})^2 = \frac{k^2}{4}$$

If $k \geq 6$, by (F3), $\delta(X) = k \geq f(\frac{k^2}{2}) = f_1(\frac{k^2}{4}) \geq f_1(|X|)$, as desired.
If $k \leq 5$, then $|X_A| + |X_B| \leq 3$. If $|X_A| = 3$, then $X_Z = \emptyset$ and $\delta(X) = 2 \cdot 4 - 4 = 4 = f_1(4)$.
If $|X_A| = 2, |X_B| = 1$, then $\delta(X) \geq \begin{cases} 2 \cdot 5 - 5 = 5 > f_1(6) > f_1(5) \\ 2 \cdot 4 - 3 = 5 > f_1(6) > f_1(5) \end{cases}$
If $|X_A| = 2, |X_B| = 0$, then $X_Z = \emptyset$ and $\delta(X) = 2 \cdot 3 - 2 = 4 = f(8) > f(6) = f_1(3)$.
If $|X_A| = 1, |X_B| = 0$, then $X_Z = \emptyset$ and $\delta(X) = 2 \cdot 2 - 1 = 3 = f(4) = f_1(2)$.
By symmetry, we see that $X \in \mathbf{K}_{f_1}$.

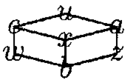
(2):Let $x_0 A \subset X \subseteq G$. We show that $\delta(X/x_0 A) > 0$. We may assume $X < G$. By † we have

$$\delta(X/x_0 A) = \delta(x_0 X_A X_B / x_0 A) = \delta(X_B / x_0 A) = \delta(X_B / x_0) = 2|X_B| - |X_B| > 0.$$

(3): It is clear that $\mathrm{cl}_G(x_0) = x_0, \mathrm{cl}_G(x_0 A) = x_0 A, \mathrm{cl}_G(x_0 B) = x_0 B$, and $\delta(A/Bx_0) = \delta(A/x_0)$. We also have $x_0 AB \leq \mathrm{cl}_G(x_0 AB) = G$, because $\delta(Z'/x_0 AB) = \sum_{z \in Z'} \delta(z/x_0 AB) = 0$. So, by Fact 4.2, we are done. $\square$

Notation 6.5. Suppose that $C_n = \{c_i : 0 \leq i \leq n\}$ and $C_n \cap A_n B_n = \emptyset$.
Let E_n be the free amalgam of $G(A_n, B_n, x_0), G(B_n, C_n, x_0)$ and $G(C_n, A_n, x_0)$.
i.e.
Edges = edges of $G(A_n, B_n, x_0), G(B_n, C_n, x_0)$ and $G(C_n, A_n, x_0)$, only.
In particular, we have $G(A_n, B_n, x_0)G(B_n, C_n, x_0) = G(A_n, B_n, x_0) \otimes_{B_n x_0} G(B_n, C_n, x_0)$, $G(B_n, C_n, x_0)G(C_n, A_n, x_0) = G(B_n, C_n, x_0) \otimes_{C_n x_0} G(C_n, A_n, x_0)$ and $G(C_n, A_n, x_0)G(A_n, B_n, x_0) = G(C_n, A_n, x_0) \otimes_{A_n x_0} G(A_n, B_n, x_0)$.

Lemma 6.6. *Suppose that $A, B, C \in \mathbf{K}_{f_1}$, $|A|, |B|, |C| \leq 4$. Suppose that $A \cap B < A, B$, $A \cap C < A, C$ and $B \cap C < B, C$, and $AB = A \otimes_{A \cap B} B, AC = A \otimes_{A \cap C} C, BC = B \otimes_{B \cap C} C$. Put $X = A \cap B \cap C, Z = A \setminus (B \cup C), W = B \setminus (A \cup C), U = C \setminus (A \cup B)$.*
Suppose that $D = ABC \notin \mathbf{K}_{f_1}$

Then D is isomorphic to , *where $a \in A \cap C, b \in A \cap B, c \in B \cap C, x \in X, z \in Z, w \in W, u \in U$.*

Proof. See Appendix 2. As $A \cap B < A$, if $c \in A \setminus (A \cap B)$, there is no $a, b \in A \cap B$ such that $R(a,c) \wedge R(b,c)$. This easy fact is important for the proof. (F1),(F2) and (F5) are also needed. □

Lemma 6.7. (1) $E_n \in \mathbf{K}_{f_1}$

(2) $E_n < E_{n+1}$, *so we may assume* $E_n < E_{n+1} < M_{f_1}$ *for any* $n < \omega$.

Proof. (1): Let $D \subseteq E_n$ and $D_{AB} = D \cap G(A_n, B_n, x_0), D_{BC} = D \cap G(B_n, C_n, x_0), D_{CA} = D \cap G(C_n, A_n, x_0)$ and $D_A = D \cap x_0 A_n, D_B = D \cap x_0 B_n, D_C = D \cap x_0 C_n$. By way of contradiction, suppose that $\delta(D) < f_1(|D|)$.

Claim. $|D_{AB}|, |D_{BC}|, |D_{CA}| \leq 4$.

Suppose that $\delta(D_{BC}), \delta(D_{CA}) \leq \delta(D_{AB}) =: d_{AB}$. Put $D' = D_{AB} D_{CA}$. By Fact 6.2, $G(A_n, B_n, x_0) G(C_n, A_n, x_0) \in \mathbf{K}_{f_1}$. So we have $D' \neq D$. As $E_n = G(A_n, B_n, x_0) G(C_n, A_n, x_0) \otimes_{B_n C_n x_0} G(B_n, C_n, x_0)$ and $B_n C_n x_0 \leq G(B_n, C_n, x_0)$ by (3) of Lemma 6.4, we see

$$D' \leq D.$$

As $x_0 A_n < G(C_n, A_n, x_0)$ (so $D_A < D_{CA}$) and $D' = D_{AB} \otimes_{D_A} D_{CA}$, so

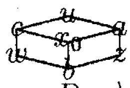
$$\delta(D') \geq d_{AB} + 1.$$

Subclaim 1: $f^{-1}(d_{AB} + 1) < 3 f^{-1}(d_{AB})$.
Note that $f^{-1}(d_{AB}) \geq f^{-1}(\delta(D_{**})) \geq 2|D_{**}|$. Suppose that this subclaim does not hold, then we have

$$f^{-1}(d_{AB} + 1) \geq 3 f^{-1}(d_{AB}) \geq 2(|D_{AB}| + |D_{BC}| + |D_{CA}|) \geq 2|D|.$$

So, we have $\delta(D) \geq \delta(D') \geq d_{AB} + 1 \geq f_1(|D|)$, a contradiction. This subclaim is proven.
Subclaim 2: $d_{AB} < f(10)$.
Otherwise, we have $f^{-1}(d_{AB}) \geq 10$. Thus, by ((F4):$f(3n) \leq f(n) + 1$), we have $3 f^{-1}(d_{AB}) \leq f^{-1}(f(f^{-1}(d_{AB})) + 1) = f^{-1}(d_{AB} + 1)$, this contradicts subclaim 1. Subclaim 2 is proven.

As $\delta(D_{**}) \leq d_{AB} < f(10)$, and $D_{**} \in \mathbf{K}_{f_1}$, we see the claim.

By this claim and Lemma 6.6, we have the following graph 

, where $a \in D_A, b \in D_B, c \in D_C, z \in D_{AB} \setminus D_A D_B, w \in D_{BC} \setminus D_B D_C, u \in D_{CA} \setminus D_A D_C$. But this is impossible by definition of E_n.
(2): Let $V = \{z_{i,n+1}, w_{i,n+1}, u_{i,n+1} : 0 \leq i \leq n\}$ be the vertices of $E_{n+1} \setminus (E_n \cup \{a_{n+1}, b_{n+1}, c_{n+1}\})$. Then

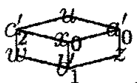
$$E_{n+1} = E_n \cup \{a_{n+1}, b_{n+1}, c_{n+1}\} \cup V.$$

Let $X \subseteq \{a_{n+1}, b_{n+1}, c_{n+1}\} \cup V$. Then $e(X, E_n) = |X|$, so $\delta(X/E_n) = \delta(X) - |X| = |X| - e(X)$. If $X \cap V = \emptyset$ or $X \cap \{a_{n+1}, b_{n+1}, c_{n+1}\} = \emptyset$, then $e(X) = 0$. Otherwise, $e(X) = |X \cap V| < |X|$, as desired. □

Theorem 6.8. *$\mathrm{Th}(M_{f_1})$ has* SOP_3.

Proof. Let $\varphi(x_1y_1z_1, x_2y_2z_2) \equiv \bigwedge_{i=1,2}(R(x_0, x_i) \wedge R(x_0, y_i) \wedge R(x_0, z_i)) \wedge \exists z, w, u(R(x_1, z) \wedge R(z, y_2) \wedge R(y_1, w) \wedge R(w, z_2) \wedge R(z_1, u) \wedge R(u, x_2))$.
Let a_n, b_n, c_n be as in E_n $(n < \omega)$, and put $d_n = a_n b_n c_n$.
Then $M_{f_1} \models \varphi(d_i, d_j)$ for $i < j < \omega$.

By way of contradiction, suppose that there exist $N \models \mathrm{Th}(M_{f_1})$ and $d_0', d_1', d_2' \in N$ such that $N \models \varphi(d_0', d_1') \wedge \varphi(d_1', d_2') \wedge \varphi(d_2', d_0')$. Let $d_i' = a_i' b_i' c_i'$.

Now we have  in N.
But any substructure of N is in $\mathbf{K}_{f_1}$, a contradiction. □

7. Appendix 1 (Free AP of $\mathbf{K}_{f_1}$)

We show Lemma 6.2, when $|X_1| \leq 6$ and

$$\frac{\delta(X_1) - \delta(X_0)}{|X_1| - |X_0|} \leq \frac{\delta(X) - \delta(X_0)}{|X| - |X_0|} \leq \frac{\delta(X_2) - \delta(X_0)}{|X_2| - |X_0|}.$$

By assumption and $|X| - |X_1| = |X_2| - |X_0|$, $\delta(X_2/X_0) \geq \delta(X_1/X_0)\dfrac{|X| - |X_1|}{|X_1| - |X_0|}$ follows.

Remark 7.1. (1) $\delta(X) \geq \delta(X_1) + \delta(X_1/X_0)\dfrac{|X| - |X_1|}{|X_1| - |X_0|}$.

(2) $f'(x)(\leq \frac{1}{14})$ is decreasing for $x \geq 14$ by (F2).

(3) $e(X_1 \setminus X_0, X_0) \leq |X_1 \setminus X_0|$ by $X_0 < X_1$. So we have

$$\delta(X_1/X_0) \geq |X_1 \setminus X_0| - e(X_1 \setminus X_0).$$

(4) X_0, X_1, X_2 do not contain 3-cycles, since they belong to $\mathbf{K}_{f_1}$.

Proof. (3): $\delta(X_1/X_0) = \delta(X_1 \setminus X_0) - e(X_1 \setminus X_0, X_0) \geq \delta(X_1 \setminus X_0) - |X_1 \setminus X_0| = |X_1 \setminus X_0| - e(X_1 \setminus X_0)$. □

Now we check $\delta(X) \geq f(2|X|)$ for each case on the size of $X_1 \setminus X_0$, X_0.
Recall (F1): $f(0) = 0, f(2) = 2, f(4) = 3, f(8) = 4 < f(10) < 4\frac{1}{2} < f(12) < 5 < f(14) < 5\frac{1}{3}$.

The case that $|X_1 \setminus X_0| = 1$

- $|X_1 \setminus X_0| = 1, |X_0| = 0$

$\delta(X) \geq 2 + 2\frac{|X|-1}{1} = 2|X| \geq f(2|X|)$.
(By $\delta(X_1) = \delta(X_1/X_0) = 2$ and $2x \geq f(2x)$ for $x \geq 2$)

- $|X_1 \setminus X_0| = 1, |X_0| = 1$

$\delta(X) \geq (4-1) + (2-1)\frac{|X|-2}{1} = 1 + |X| \geq f(2|X|)$.
(By $1 + x \geq f(2x)$ and $\delta(X_1) \geq 4-1, \delta(X_1/X_0) \geq 2-1$.)

- $|X_1 \setminus X_0| = 1, |X_0| = 2$

$\delta(X) \geq (6-2) + 1\frac{|X|-3}{1} = 1 + |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 6-2, \delta(X_1/X_0) \geq 2-1$ and $1 + x \geq f(2x)$)

- $|X_1 \setminus X_0| = 1, |X_0| = 3$

$\delta(X) \geq (8-3) + 1\frac{|X|-4}{1} = 1 + |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 8-3, \delta(X_1/X_0) \geq 2-1$ and $1 + x \geq f(2x)$)

- $|X_1 \setminus X_0| = 1, |X_0| = 4$

$\delta(X) \geq (10-5) + 1\frac{|X|-5}{1} = 1 + |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 10-5, \delta(X_1/X_0) \geq 2-1$ and $x \geq f(2x)$ if $x \geq 6$.)

- $|X_1 \setminus X_0| = 1, |X_0| = 5$

$\delta(X) \geq (12-6) + 1\frac{|X|-6}{1} = |X| \geq f(2|X|)$.
(By $\delta(X_1) \geq 12-6, \delta(X_1/X_0) \geq 2-1$ and $x \geq f(2x)$ if $x \geq 6$.)

The case that $|X_1 \setminus X_0| = 2$

- $|X_1 \setminus X_0| = 2, |X_0| = 0$

$\delta(X) \geq 3 + 3\frac{|X|-1}{1} \geq f(2|X|)$.
(By $\delta(X_1) = \delta(X_1/X_0) \geq 3$ and $3x + 2 \geq f(2x)$.)

- $|X_1 \setminus X_0| = 2, |X_0| = 1$

$\delta(X) \geq 4 + 2\frac{|X|-3}{2} \geq f(2|X|)$.
(By $\delta(X_1) \geq 6-2, \delta(X_1/X_0) \geq 3-1$ and $x + 1 \geq f(2x)$.)

- $|X_1 \setminus X_0| = 2, |X_0| = 2$

$\delta(X) \geq 4 + 1\frac{|X|-4}{2} \geq f(2|X|)$.
(As $\delta(X_1) \geq 8-4, \delta(X_1/X_0) \geq 3-2$ and $4 + \frac{x-4}{2} \geq f(2x)$ if $x \geq 5$.)

- $|X_1 \setminus X_0| = 2, |X_0| = 3$

$\delta(X) \geq 5 + 1\frac{|X|-5}{2} \geq f(2|X|)$.
(As $\delta(X_1) \geq 10-5, \delta(X_1/X_0) \geq 3-2$ and $5 + \frac{x-5}{2} \geq f(2x)$ if $x \geq 6$.)

- $|X_1 \setminus X_0| = 2, |X_0| = 4$

$\delta(X) \geq 5 + 1\frac{|X|-6}{2} \geq f(2|X|)$.
(As $\delta(X_1) \geq 12-7, \delta(X_1/X_0) \geq 3-2$ and $5 + \frac{x-6}{2} \geq f(2x)$ if $x \geq 7$.)

The case that $|X_1 \setminus X_0| = 3$

- $|X_1 \setminus X_0| = 3, |X_0| = 0$

$\delta(X) \geq 4 + 4\frac{|X|-3}{3} \geq f(2|X|)$.
(As $\delta(X_1) = \delta(X_1/X_0) \geq 6-2$ and $4 + 4\frac{x-3}{3} \geq f(2x)$ if $x \geq 4$.)

- $|X_1 \setminus X_0| = 3, |X_0| = 1$

$\delta(X) \geq 5 + 3\frac{|X|-4}{3} \geq f(2|X|)$.
(As $\delta(X_1) \geq 8-3$, $\delta(X_1/X_0) \geq 4-1$ and $5 + 3\frac{x-4}{3} = x+1 \geq f(2x)$ if $x \geq 5$.)

- $|X_1 \setminus X_0| = 3, |X_0| = 2$

$\delta(X) \geq 5 + 2\frac{|X|-5}{3} \geq f(2|X|)$.
(As $\delta(X_1) \geq 10-5$, $\delta(X_1/X_0) \geq 4-2$ and $5 + 2\frac{x-5}{3} \geq f(2x)$ if $x \geq 6$.)

- $|X_1 \setminus X_0| = 3, |X_0| = 3$

$\delta(X) \geq 5 + 1\frac{|X|-6}{3} \geq f(2|X|)$.
(As $\delta(X_1) \geq 12-7$, $\delta(X_1/X_0) \geq 4-3$ and $5 + 1\frac{x-6}{3} \geq f(2x)$ if $x \geq 7$.)

The case that $|X_1 \setminus X_0| = 4$

- $|X_1 \setminus X_0| = 4, |X_0| = 0$

$\delta(X) \geq 4 + 4\frac{|X|-4}{4} \geq f(2|X|)$.
(As $\delta(X_1) = \delta(X_1/X_0) \geq 8-4$ and $4 + 4\frac{x-4}{4} = x \geq f(2x)$ if $x \geq 5$.)

- $|X_1 \setminus X_0| = 4, |X_0| = 1$

$\delta(X) \geq 5 + 3\frac{|X| - 5}{4} \geq f(2|X|)$.
(As $\delta(X_1) \geq 10 - 5$, $\delta(X_1/X_0) \geq 4 - 1$ and $5 + 3\frac{x - 5}{4} \geq f(2x)$ if $x \geq 6$.)

- $|X_1 \setminus X_0| = 4, |X_0| = 2$

$\delta(X) \geq 5 + 2\frac{|X| - 6}{4} \geq f(2|X|)$.
(As $\delta(X_1) \geq 12 - 7$, $\delta(X_1/X_0) \geq 4 - 2$ and $5 + \frac{x - 6}{2} \geq f(2x)$ if $x \geq 7$.)

The case that $|X_1 \setminus X_0| = 5$

- $|X_1 \setminus X_0| = 5, |X_0| = 0$

$\delta(X) \geq 5 + 5\frac{|X| - 5}{5} = |X| \geq f(2|X|)$.
(As $\delta(X_1) = \delta(X_1/X_0) \geq 10 - 5$ and $x \geq f(2x)$ if $x \geq 6$.)

- $|X_1 \setminus X_0| = 5, |X_0| = 1$

$\delta(X) \geq 5 + 3\frac{|X| - 6}{5} \geq f(2|X|)$.
(As $\delta(X_1) \geq 12 - 7$, $\delta(X_1/X_0) \geq 5 - 2$ and $5 + 3\frac{x - 6}{5} \geq f(2x)$ if $x \geq 7$.)

The case that $|X_1 \setminus X_0| = 6$

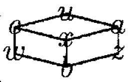
- $|X_1 \setminus X_0| = 6, |X_0| = 0$

$\delta(X) \geq f(12) + f(12)\frac{|X| - 6}{6} = f(12)|X| \geq f(2|X|)$.
(As $\delta(X_1) = \delta(X_1/X_0) \geq f(12)$ and $f(12)|X| > 4|X| \geq f(2x)$ if $x \geq 7$.) $\square$

8. Appendix 2 (The proof of Lemma 6.6)

We show the following.

Lemma 6.6 *Suppose that $A, B, C \in \mathbf{K}_{f_1}$, $|A|, |B|, |C| \leq 4$. And suppose that $A \cap B < A, B$, $A \cap C < A, C$ and $B \cap C < B, C$, and $AB = A \otimes_{A \cap B} B, AC = A \otimes_{A \cap C} C, BC = B \otimes_{B \cap C} C$. Put $X = A \cap B \cap C, Z = A \setminus (B \cup C), W = B \setminus (A \cup C), U = C \setminus (A \cup B)$.*

If $D = ABC \notin \mathbf{K}_{f_1}$, then D is isomorphic to  *, where $a \in A \cap C, b \in A \cap B, c \in B \cap C, x \in X, z \in Z, w \in W, u \in U$.*

Proof. We use the following easy fact: If $X < Y$, $c \in Y \setminus X, a, b \in X$, then $R(a, c) \wedge R(b, c)$ does not hold.

Clearly, $D = BCZ$.
We may assume that $Z, W, U \neq \emptyset$, since, for example, if $Z = \emptyset$, then $D = B \otimes_{B\cap C} C \in \mathbf{K}_{f_1}$ by free AP. As $|A|, |B|, |C| \leq 4$, we have $|A \cap C| \leq 3$.

a, a' denote elements of $A \cap C$, b, b' denote elements of $A \cap B$, c, c' denote elements of $B \cap C$, z, z' denote elements of Z, w, w', w'' denote elements of W, u, u' denote elements of U and x, x' denote elements of X.

We check each case on the size of $|A \cap C|$.

The case that $|A \cap C| = 3$

We have $6 \leq |D| \leq 9$. As $|A| \leq 4$, $|Z| = 1$ and $A \cap B \setminus X = \emptyset$ follow. So, we have $\delta(Z/BC) \geq 1$. Thus $\delta(D) = \delta(BC) + \delta(Z/BC) \geq f(2|D| - 2) + 1 \geq f(2|D|)$.

The case that $|A \cap C| = 2$

- $|(A \cap C) \setminus X| = 2$ (i.e. $X = \emptyset$.)

Suppose that $|Z| = 2$. So, $6 \leq |D| \leq 10$.
As $A \cap B = \emptyset$, $\delta(Z/BC) \geq 3 - 2$ follows. So, $\delta(D) \geq \delta(BC) + 1 \geq f(2|D| - 4) + 1 \geq f(2|D|)$ by (F5),$f(8) + 1 = 5 \geq f(12), f(14) + 1 \geq 6 \geq f(18)$ and (F2).

Suppose that $|Z| = 1$, so $|A \cap B| \leq 1$.

If $A \cap B = \emptyset$, then $5 \leq |D| \leq 9$, $\delta(Z/BC) \geq 2 - 1$ follows. So, $\delta(D) \geq \delta(BC) + 1 \geq f(2|D| - 2) + 1 \geq f(2|D|)$.

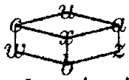
If $|A \cap B| = 1$, then $6 \leq |D| \leq 9$.
If $|D| = 6$, then $D = aa'zbwu$. Then $\delta(D) = 12 - 5 = 7 \geq f(12)$.
If $|D| = 7$, then $D = aa'zbwuu'$, $aa'zbwcu$ or $aa'zbww'u$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 14 - 7 = 7 \geq f(14)$.
If $|D| = 8$, then $D = aa'zbww'uu'$ or $aa'zbww'cu$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 16 - 9 = 7 \geq f(16)$.
If $|D| = 9$, then $D = aa'zbww'w''uu'$, because $Z, W, U \neq \emptyset$. Then $\delta(D) \geq 18 - 9 = 9 \geq f(18)$.

- $|(A \cap C) \setminus X| = |X| = 1$.

Suppose that $|A \cap B \setminus X| = 0$. Then $\delta(Z/BC) \geq 1$.
So, $\delta(D) \geq f(2|D| - 2|Z|) + 1$.

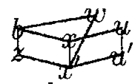
If $|Z| = 1$, then $5 \leq |D| \leq 8$, so $f(2|D|-2)+1 \geq f(2|D|)$ holds.
If $|Z| = 2$, then $6 \leq |D| \leq 9$. $f(2|D|-4)+1 \geq f(2|D|)$ holds for $|D| = 6, 9$. ($f(8)+1 = 5 \geq f(12)$ and $f(14)+1 \geq 6 \geq f(18)$.) For $|D| = 7$, $D = xazz'wcu$, $xazz'wuu'$ or $xazz'ww'u$ and then $\delta(D) \geq 14-8 \geq f(14)$ holds. For $|D| = 8$, $D = xazz'ww'cu$ or $xazz'ww'uu'$ and then $\delta(D) \geq 16-10 \geq f(16)$ holds.

Suppose that $|A \cap B \setminus X| = 1$. Then $6 \leq |D| \leq 8$.
If $|D| = 6$, then $D = xazbwu$ and $\delta(D) \geq 12-6 \geq f(12)$.
If $|D| = 7$, then $D = xazbwuu'$, $xazbww'u$ or $xazbwcu$. If the former two cases hold, then $\delta(D) \geq 14-8 \geq f(14)$.

In the latter case, D is  if and only if $\delta(D) = 14-9 < f(14)$.
If $|D| = 8$, then $D = xazbww'uu'$ and $\delta(D) \geq 16-10 \geq f(16)$.

- $|(A \cap C) \setminus X| = 0, |X| = 2$

We have $5 \leq |D| \leq 8$.
If $|D| = 5$, then $D = xx'zwu$ and $\delta(D) \geq 10-4 \geq f(10)$.
If $|D| = 6$, then $D = xx'zz'wu$, $xx'zbwu$, $xx'zww'u$, $xx'zwcu$ or $xx'zwuu'$ and $\delta(D) \geq 12-7 \geq f(12)$.
If $|D| = 7$, then $D = xx'zz'ww'u$, $xx'zbwuu'$, $xx'zww'uu'$, $xx'zz'wcu$, $xx'zz'wuu'$, $xx'zbwuu'$ or $xx'zww'uu'$.
If $D \neq xx'zz'wcu, xx'zbwuu'$, then $\delta(D) \geq 14-8 \geq f(14)$. And we have

$\delta(D) = 14-9 < f(14)$ if and only if D is  or . But this never happens, because $B \cap C < B$ and $A \cap B < B$, so w does not have two edges to $B \cap C$, also to $A \cap B$.
If $|D| = 8$, then $D = xx'zz'ww'uu'$ and $\delta(D) = 16-10 \geq f(16)$.

The case that $|A \cap C| = 1$

- $|(A \cap C) \setminus X| = 1$ ($|X| = 0$)

By symmetry, we may assume $|A \cap B|, |B \cap C| \leq 1$.

Suppose that $|A \cap B|, |B \cap C| = 1$. Then $6 \leq |D| \leq 9$.
If $|D| = 6$, $\delta(D) \geq 12-6 \geq f(12)$. If $|D| = 7$, $\delta(D) \geq 14-7 \geq f(14)$. If $|D| = 8$, $\delta(D) \geq 16-8 \geq f(16)$. If $|D| = 9$, $\delta(D) \geq 18-9 \geq f(18)$.

Suppose that $|A \cap B| = 0$ or $|B \cap C| = 1$.
By symmetry, we assume that $|A \cap B| = 0$. Then $AC \cap B = B \cap C$. By assumption on A, B, C, $B \cap C < C < AC$ and $AC = A \otimes_{A \cap C} C \in \mathbf{K}_{f_1}$ by free AP. As $B \cap C < AC, B$ and $D = AC \otimes_{A \cap C} B$, we have $D \in \mathbf{K}_{f_1}$ by free AP.

- $|(A \cap C) \setminus X| = 0$ and $|X| = 1$.

As we have shown tha case that $|A \cap C| = 2, 3$, by symmetry, we may assume that $D = XZWU$. (i.e. $|(A \cap B) \setminus X| = 0$ and $|(B \cap C) \setminus X| = 0$) As $X < XZW = XZ \otimes_X XW \in \mathbf{K}_{f_1}$ and $X < XU \in K_{f_1}$, we have $D = XZW \otimes_X XZ \in \mathbf{K}_{f_1}$ by free AP.

The case that $|A \cap C| = 0$

As we have shown tha case that $|A \cap C| = 1, 2, 3$, by symmetry, we may assume that $D = ZWU$. (i.e. $|A \cap B| = 0$ and $|B \cap C| = 0$.) By free AP, we see $D \in \mathbf{K}_{f_1}$. □

References

[E] D.M.Evans, Some remarks on generic structures, preprint, July 2003 (working draft)

[E1] D.M.Evans, $\aleph_0$-categorical structures with a predimensions, Annals of Pure and Applied Logic 120 (2003) 157-186.

[P] M.Pourmahdian, Smooth classes without AC and Robinson theories, The Journal of Symbolic Logic 67 (2002) 1274-1294.

[Sh] S.Shelah, Toward classifying unstable theories, Annals of Pure and Applied Logic 80 (1996) 229-255.

E-mail address: `ikuo.yoneda@s3.dion.ne.jp`